Simple proof of a determinant equation

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Lemma: $det(T) = \prod_{k=0}^{n} a_{kk}$ for an upper or lower triangular matrix T^{nxn}.

Proof: We use proof by induction. Our base case is $T^{2x2} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$. Then, $\det(T^{2x2}) = a_{11}a_{22}$. We assume that $det(T^{(n-1)x(n-1)}) = \prod_{i}^{r} a_{ii}$ By Laplace expansion along the first column, $\det(T^{nxn}) = a_{11} det \begin{pmatrix} a_{22} & 0 & \dots & a_{2n} \\ 0 & a_{33} & \dots & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & a_{nn} \end{pmatrix}$. The matrix collapses in the same manner during each iteration. Therefore, the proof by induction holds. The proof for a lower triangular matrix follows the same procedure.

Prop: The determinant of an invertible matrix A is equal to the product of its eigenvalues. We write this as $det(A) = \prod_{i}^{r} \lambda_i$ where $r = rank(A)$ and λ satisfies $Ax = \lambda x$ for an eigenvector x.

Proof: Consider the matrix $A = [a_{ij}] \in \Re^{nxm}$. If A is diagonalizable, we have the eigendecomposition $A = E\Lambda E^{-1}$ where E is a matrix of r linearly independent eigenvectors of A, E^{-1} is its inverse, and Λ is a diagonal matrix holding the eigenvalues of A. $M_n(\Re) \xrightarrow{\det} \Re$ is a multiplicative morphism; thus, $det(A \cdot B) = det(A) \cdot det(B)$. We apply this fact to the eigendecomposition of A and find that $det(A) = det(E) \cdot det(\Lambda) \cdot det(E^{-1})$. Since $det(E) = \frac{1}{det(E^{-1})}$, we find that $det(A) = det(\Lambda)$. As a diagonal matrix (considered upper or lower triangular, by definition), $det(\Lambda)$ is simply the product of elements on the diagonal, as demonstrated by the Lemma. Therefore, $det(A) = \prod_{i}^{r} \lambda_i$. Q.E.D.